

Isometry classification of cubic homogeneous 3-dimensional forms

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Abstract

The problem of classification of cubic homogeneous Finslerian 3D metrics with respect to their isometries is considered. It is shown, that there are 6 different general affine types of such metrics. Algebras of isometries are presented in apparent kind together with their affine-invariant properties. Interrelation between symmetries and projective classifings is discussed.

1 Introduction

One of the main tools for studying invariant geometrical properties of manifolds is symmetry considered in a wide sense of the word. The most important and simple kind of symmetry is isometry, which can be defined for any manifold with metric G . Local definition of continuous (or, more correctly, smooth) isometry involves vector field X (*Killing field*), satisfying the following *Killing equation*:

$$L_X G = 0, \tag{1}$$

where L_X denotes standard Lie derivative along X [1]. It is well known, that isometries fields form Lie algebra with respect to Lie bracket, and invariant properties of the Lie algebra (dimension, solvability, presence or absence of subalgebras and ideals etc.) express invariant properties of the manifolds itself. Often Lie algebra of isometries defines metrics G uniquely or up to some arbitrary functions. For example, homogeneous space-times in cosmological models of GR admits complete Bianchi classification, which is convenient mean for classifying both of the models itself and their important physical properties [2, 3].

Isometries of manifolds with homogeneous quadratic metrics (i.e. satisfying $G_{\alpha\beta} = \text{const}$), are well known. If we exclude degenerate cases, we always deals with metrics of: a) euclidean type; b) pseudo-euclidean type; c) symplectic type d) mixed type, containing all or part of the types (a), (b), (c) in the form of their direct sum. In the case (a) we have isometry groups $O(n)$, in the case (b) — groups $O(m, n)$, in the case (c) — groups $Sp(n)$ for some natural n, m . In case (d) we have some combinations of above listed groups. These types of metrics together with their groups of isometries form local geometric base for construction of more complicated physical models of space-time, relevant to some modern physical concepts (say, gauge principle) and experimental data.

Last decade we observe increasing interest both from physics and from mathematics for geometrical models involving nonquadratic metrics of Finslerian kind [4, 5, 6]. Such

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models are of great interest, since they naturally reflect possible anisotropy of space-time. The hypothesis about anisotropy of space-time makes more clear some experimental data in cosmology, astrophysics and elementary particles physics.

In view of further development of Finslerian geometrical models we need more deeper understanding of their geometrical properties. Present paper is devoted to investigation of what we can know about homogeneous cubic metrics by methods of Lie isometries theory. We restrict ourselves by cubic metrics in 3D space, which play important role for hypercomplex numbers¹. Our analysis shows, that in a difference with isometries of quadratic homogeneous metrics in 3D 1) isometries of homogeneous cubic metrics form more rich family (for nondegenerate metrics 5 cubic against 3 quadratic); 2) isometry classification is not complete, since some different projective classes of cubic metrics belong to the same symmetry class (see table at the end of the section 5).

These results (obtained for the simplest class of Finslerian metrics) at least show, that classical Lie analysis is useful but insufficient for relevant understanding of Finslerian models and more subtle aspects of symmetry should be incorporated in this topic.

2 Metrics and affine types

We are going to investigate isometries of homogenous cubic metrics of the form:

$$G = G_{\alpha\beta\gamma} dx^\alpha \otimes dx^\beta \otimes dx^\gamma, \quad (2)$$

where $G_{\alpha\beta\gamma}$ — symmetric real cubic matrix. Geometrical spaces with metric of such type are commonly referred to *Finslerian spaces* and metrics G is commonly related to special *Finslerian metric* [7]. Let us introduce the following notations:

$$G_{\alpha\alpha\alpha} = A_\alpha; \quad G_{122} = B_1; \quad G_{133} = B_2; \quad G_{233} = B_3; \quad (3)$$

$$G_{112} = C_1; \quad G_{113} = C_2; \quad G_{223} = C_3; \quad G_{123} = F,$$

where all A_α , B_β , C_γ and F are constants.

Not all of the components (3) have geometric significance. Representation (3) is invariant with respect to choice of coordinate systems within class of affine-equivalent ones, where all components of G are constant. Any matrix of nondegenerate affine homogeneous transformation in R^3 has in general 9 independent components, which could be used so, that 9 from 10 components of G will vanish.

So, we can preliminary conclude, that:

1. For complete investigation of the problem it is sufficient to consider metrics with some small number of nonzero components;
2. It is necessary to investigate all possible combinations of these components.

We will show, that it is, in fact, sufficiently to study the metrics G with nonzero components of number no greater than 6.

The number of nonzero coefficients of homogeneous metric G will define *its affine type* $\tau(G)$. Note, that affine type of metric G depends on choice of affine coordinate system. Invariant characteristic, independent on choice of affine coordinates, is *exact affine type*:

$$\tau_0(G) \equiv \min_{\text{Aff}(R^3)} \tau(G),$$

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where $\text{Aff}(R^3)$ — class of affine coordinate system in R^3 , connected by nondegenerate affine transformations. Lets call two homogeneous metrics G_1 and G_2 *equivalent*: $G_1 \sim G_2$, if there exist such homogeneous nondegenerate affine transformation in R^3 , which transforms G_1 into G_2 or vice verse. Obviously, for equivalent metrics: $G_1 \sim G_2$ may be $\tau(G_1) \neq \tau(G_2)$, but it is necessarily must be: $\tau_0(G_1) = \tau_0(G_2)$. However, coincidence of exact affine types for some two metrics, generally speaking, is not sufficient for their equivalence, since the components, which compose minimal sets of nonzero ones may be different for these two metrics.

3 Killing equations and their solutions

General system of Killing equations (1) takes the form:

$$\left\{ \begin{array}{l} 3A_1\partial_1X^1 + 3C_1\partial_1X^2 + 3C_2\partial_1X^3 = 0; \\ 2C_1\partial_1X^1 + 2B_1\partial_1X^2 + 2F\partial_1X^3 + A_1\partial_2X^1 + C_1\partial_2X^2 + C_2\partial_2X^3 = 0; \\ B_1\partial_1X^1 + A_2\partial_1X^2 + C_3\partial_1X^3 + 2C_1\partial_2X^1 + 2B_1\partial_2X^2 + 2F\partial_2X^3 = 0; \\ 3B_1\partial_2X^1 + 3A_2\partial_2X^2 + 3C_3\partial_2X^3 = 0; \\ 2C_2\partial_1X^1 + 2F\partial_1X^2 + 2B_2\partial_1X^3 + A_1\partial_3X^1 + C_1\partial_3X^2 + C_2\partial_3X^3 = 0; \\ F\partial_1X^1 + C_3\partial_1X^2 + B_3\partial_1X^3 + C_2\partial_2X^1 + F\partial_2X^2 + B_2\partial_2X^3 + C_1\partial_3X^1 + \\ + B_1\partial_3X^2 + F\partial_3X^3 = 0; \\ 2F\partial_2X^1 + 2C_3\partial_2X^2 + 2B_3\partial_2X^3 + B_1\partial_3X^1 + A_2\partial_3X^2 + C_3\partial_3X^3 = 0; \\ B_2\partial_1X^1 + B_3\partial_1X^2 + A_3\partial_1X^3 + 2C_2\partial_3X^1 + 2F\partial_3X^2 + 2B_2\partial_3X^3 = 0; \\ B_2\partial_2X^1 + B_3\partial_2X^2 + A_3\partial_2X^3 + 2F\partial_3X^1 + 2C_3\partial_3X^2 + 2B_3\partial_3X^3 = 0; \\ 3B_2\partial_3X^1 + 3B_3\partial_3X^2 + 3A_3\partial_3X^3 = 0. \end{array} \right. \quad (4)$$

Let us consider consequently all cases of general metrics with different $\tau(G)$. Everywhere we'll use freedom of scales of coordinates for transforming of maximal number of components into ± 1 (canonical kind). We'll not consider separately those of the cases, which differ from each other by permutations of coordinates. Also we'll omit constant vector fields of isometries, forming subalgebra of translations of complete algebra of isometries of G , and will focus only on symmetries, different from translations. We shall call them *nontrivial symmetries* of homogeneous Finslerian metrics.

3.1 Metrics with $\tau(G) = \tau_0(G) = 1$ (3 types)

In notation of different cases only nonzero components of metric are shown (all remaining are zero). Hereafter we list only the cases with nontrivial symmetries.

1. $F \neq 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^3), \quad (5)$$

where \hat{S} — tensor product symmetrization operator. This metric is known as *Berwald-Moor metric*. Nontrivial symmetries are [8, 9]:

$$X_1 = x^1\partial_1 - x^2\partial_2; \quad X_2 = x^1\partial_1 - x^3\partial_3.$$

These are unimodular dilatations of coordinate axes. Note, that this algebra is abelian.

2. $B_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^2 \otimes dx^2 + dx^2 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^2 \otimes dx^1. \quad (6)$$

Algebra of nontrivial symmetries is infinitely dimensional:

$$X = x^2\partial_2 - 2x^1\partial_1 + f(x^1, x^2, x^3)\partial_3,$$

where f — arbitrary smooth function of three variables.

3. $A_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1. \quad (7)$$

Algebra of nontrivial symmetries is infinitely dimensional:

$$X = f_2(x^1, x^2, x^3)\partial_2 + f_3(x^1, x^2, x^3)\partial_3,$$

where f_2, f_3 — arbitrary smooth functions of three variables.

These three cases exhaust nontrivial cases of the class $\tau_0(G) = 1$. Note, that metrics (6)-(7) are degenerated, since they are described by subspaces of 3-dimensional basis of 1-forms $\{dx^1, dx^2, dx^3\}$.

3.2 Metrics with $\tau(G) = 2$ (9 types)

1. $F \neq 0, A_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + \hat{S}(dx^1 \otimes dx^2 \otimes dx^3). \quad (8)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^2\partial_2 - x^3\partial_3.$$

2. $F \neq 0, B_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^2 \otimes dx^2 + dx^2 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^2 \otimes dx^1 + \hat{S}(dx^1 \otimes dx^2 \otimes dx^3). \quad (9)$$

Algebra of nontrivial symmetries is 2-dimensional:

$$X_1 = x^1\partial_1 - (x^3 + x^2/2)\partial_3; \quad x^2\partial_2 - (x^3 + x^2)\partial_3.$$

3. $A_1 \neq 0, B_3 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + dx^2 \otimes dx^3 \otimes dx^3 + dx^3 \otimes dx^2 \otimes dx^3 + dx^3 \otimes dx^3 \otimes dx^2. \quad (10)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^2\partial_2 - (x^3/2)\partial_3.$$

4. $A_1 \neq 0, C_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + dx^1 \otimes dx^1 \otimes dx^3 + dx^1 \otimes dx^3 \otimes dx^1 + dx^3 \otimes dx^1 \otimes dx^1. \quad (11)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^1\partial_1 - (2x^2 + x^1)\partial_2.$$

5. $B_1 \neq 0, B_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^2 \otimes dx^2 + dx^2 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^2 \otimes dx^1 \\ \pm (dx^1 \otimes dx^3 \otimes dx^3 + dx^3 \otimes dx^1 \otimes dx^3 + dx^3 \otimes dx^3 \otimes dx^1). \quad (12)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^1\partial_1 - (x^2/2)\partial_2 - (x^3/2)\partial_3.$$

6. $B_1 \neq 0, B_3 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^2 \otimes dx^2 + dx^2 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^2 \otimes dx^1 + dx^2 \otimes dx^3 \otimes dx^3 + dx^3 \otimes dx^3 \otimes dx^2 + dx^3 \otimes dx^2 \otimes dx^3. \quad (13)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X_1 = -2x^1 \partial + x^2 \partial_2 - (x^3/2) \partial_3; \quad X_2 = x^3 \partial_1 - (x^2/2) \partial_3.$$

7. $B_1 \neq 0, C_3 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^2 \otimes dx^2 + dx^2 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^2 \otimes dx^1 + dx^3 \otimes dx^2 \otimes dx^2 + dx^2 \otimes dx^3 \otimes dx^2 + dx^2 \otimes dx^2 \otimes dx^3. \quad (14)$$

Algebra of nontrivial symmetries is ∞ -dimensional:

$$X_1 = x^2 \partial_2 - 2(x^1 + x^3) \partial_3; \quad X_2 = f(x^1, x^2, x^3)(\partial_1 - \partial_3),$$

where f — arbitrary smooth function of three variables.

8. $A_1 \neq 0, A_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \otimes dx^2 \quad (15)$$

Algebra of nontrivial symmetries is ∞ -dimensional:

$$X = f(x^1, x^2, x^3) \partial_3,$$

where f — arbitrary smooth function of three variables.

9. $A_1 \neq 0, B_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 \pm \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) \quad (16)$$

Algebra of nontrivial symmetries is ∞ -dimensional:

$$X = f(x^1, x^2, x^3) \partial_3,$$

where f — arbitrary smooth function of three variables.

3.3 Metrics with $\tau(G) = 3$ (13 types)

In majority of the cases symmetries are trivial. Only 13 metrics possess nontrivial symmetries.

1. $F \neq 0, A_1 \neq 0, B_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 \pm \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3). \quad (17)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^2 \partial_2 - (x^3 \pm x^2) \partial_3.$$

2. $F \neq 0, A_1 \neq 0, C_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3). \quad (18)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^2 \partial_2 - (x^3 + x^1/2) \partial_3.$$

3. $F \neq 0, B_1 \neq 0, B_2 \neq 0$. Canonical form of metric:

$$G = \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) \pm \hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3). \quad (19)$$

Algebra of nontrivial symmetries is 2-dimensional:

$$X_1 = x^1 \partial_1 \pm (x^3/2) \partial_2 - (x^3 + x^2/2) \partial_3; \quad X_2 = (x^2 \pm x^3) \partial_2 - (x^2 + x^3) \partial_3.$$

4. $F \neq 0, B_1 \neq 0, B_3 \neq 0$. Canonical form of metric:

$$G = \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^2 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3). \quad (20)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^1 + x^3) \partial_1 - (x^3 + x^2/2) \partial_3.$$

5. $F \neq 0, B_1 \neq 0, C_1 \neq 0$. Canonical form of metric:

$$G = \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3). \quad (21)$$

Algebra of nontrivial symmetries is 2-dimensional:

$$X_1 = x^2 \partial_2 - (x^2 + x^3) \partial_3; \quad X_2 = x^1 \partial_1 - (x^3 + x^2/2 + x^1) \partial_3.$$

6. $F \neq 0, B_1 \neq 0, C_3 \neq 0$. Canonical form of metric:

$$G = \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^2 \otimes dx^2 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3). \quad (22)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 + x^2/2) \partial_3 - (x^1 + x^2/2) \partial_1.$$

7. $A_1 \neq 0, A_2 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \otimes dx^2 + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (23)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^1 \partial_1 - (x^1 + 2x^3) \partial_3.$$

8. $A_1 \neq 0, B_1 \neq 0, B_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + \epsilon_1 \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \epsilon_2 \hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3), \quad (24)$$

where $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ — independent sign factors. Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^3 \partial_2 - \epsilon_1 \epsilon_2 x^2 \partial_3.$$

9. $A_1 \neq 0, B_1 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 \pm \hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3), \quad (25)$$

Algebra of nontrivial symmetries is 2-dimensional:

$$X_1 = x^1 \partial_1 - (x^2/2) \partial_2 - 2x^3 \partial_3; \quad X_2 = \mp (x^1/2) \partial_2 + x^2 \partial_3.$$

10. $A_1 \neq 0, B_1 \neq 0, C_3 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 \pm \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{S}(dx^2 \otimes dx^2 \otimes dx^3), \quad (26)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^2 \partial_2 - 2(x^3 \pm x^1) \partial_3.$$

11. $B_1 \neq 0, B_2 \neq 0, C_1 \neq 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) \pm \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{S}(dx^1 \otimes dx^1 \otimes dx^2), \quad (27)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^3 \partial_2 \mp 2(x^1/2 + x^2) \partial_3.$$

12. $B_1 \neq 0, B_3 \neq 0, C_1 \neq 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{S}(dx^2 \otimes dx^3 \otimes dx^3) + \hat{S}(dx^1 \otimes dx^1 \otimes dx^2). \quad (28)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^3 \partial_1 - (x^1 + x^2/2) \partial_3.$$

13. $B_1 \neq 0, B_3 \neq 0, C_3 \neq 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) \pm \hat{S}(dx^2 \otimes dx^3 \otimes dx^3) + \hat{S}(dx^2 \otimes dx^2 \otimes dx^3). \quad (29)$$

Algebra of nontrivial symmetries is 2-dimensional:

$$X_1 = x^2 \partial_2 - (x^3/2) \partial_3 - (2x^1 + 3x^3/2) \partial_1; \quad X_2 = x^2 \partial_3 - (x^2 \pm 2x^3) \partial_1.$$

3.4 Metrics with $\tau(G) = 4$ (10 types)

1. $F \neq 0, A_1 \neq 0, B_1 \neq 0, B_2 \neq 0$. Canonical form of metric:

$$G = F \hat{S}(dx^1 \otimes dx^2 \otimes dx^3) + \epsilon_1 \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) + \epsilon_2 \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) + dx^1 \otimes dx^1 \otimes dx^1. \quad (30)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 + \epsilon_2 F x^2) \partial_2 - \epsilon_2 (\epsilon_1 x^2 + F x^3) \partial_3.$$

2. $F \neq 0, A_1 \neq 0, B_1 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = F \hat{S}(dx^1 \otimes dx^2 \otimes dx^3) \pm \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{S}(dx^1 \otimes dx^1 \otimes dx^3) + dx^1 \otimes dx^1 \otimes dx^1. \quad (31)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 \pm x^2/F) \partial_3 - (x^2 + x^1/2F) \partial_2.$$

3. $F \neq 0, B_1 \neq 0, B_2 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = F \hat{S}(dx^1 \otimes dx^2 \otimes dx^3) \pm \hat{S}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{S}(dx^1 \otimes dx^1 \otimes dx^3). \quad (32)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^2 \pm F x^3) \partial_3 \mp (F x^2 + x^3 + x^1/2) \partial_2.$$

4. $F \neq 0, B_2 \neq 0, B_3 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = F\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^2 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (33)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^1 + x^3/2F)\partial_1 - (x^2 + x^1/F + x^3/2F)\partial_2.$$

5. $A_1 \neq 0, A_2 \neq 0, B_1 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \otimes dx^2 + B\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (34)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^1\partial_1 - Bx^1\partial_2 + 2(B^2x^2 - x^3 - x^1/2)\partial_3.$$

6. $A_1 \neq 0, A_2 \neq 0, B_1 \neq 0, C_3 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \otimes dx^2 + B\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^2 \otimes dx^2 \otimes dx^3). \quad (35)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^2\partial_2 - (2x^3 + 2Bx^1 + x^2)\partial_3.$$

7. $A_1 \neq 0, B_1 \neq 0, B_2 \neq 0, C_1 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + \epsilon_1\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \epsilon_2\hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3) + C\hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (36)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^3\partial_2 - (\epsilon_1\epsilon_2x^2 + \epsilon_2Cx^1/2)\partial_3.$$

8. $A_1 \neq 0, B_2 \neq 0, B_3 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 \pm \hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^2 \otimes dx^3 \otimes dx^3) + C\hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (37)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = x^3\partial_3 - Cx^3\partial_1 + (\pm Cx^3 + 2(C^2 \mp 1)x^1 - 2x^2)\partial_2.$$

9. $A_1 \neq 0, B_1 \neq 0, C_1 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = dx^1 \otimes dx^1 \otimes dx^1 + B\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (38)$$

Algebra of nontrivial symmetries is 2-dimensional:

$$X_1 = x^1\partial_1 - (x^2/2)\partial_2 - (2x^3 + x^1 + 3x^2/2)\partial_3; \quad X_2 = x^1\partial_2 - (x^1 + 2Bx^2)\partial_3.$$

10. $B_1 \neq 0, B_2 \neq 0, C_1 \neq 0, C_2 \neq 0$. Canonical form of metric:

$$G = B\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^2) + \hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3). \quad (39)$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 + x^1/2)\partial_2 - (Bx^2 + x^1/2)\partial_3.$$

3.5 Metrics with $\tau(G) = 5$ (5 types)

From the technical viewpoint this case is the most complicated, since it includes the largest number of cases under consideration. This complexity is compensated by rareness of the cases with nontrivial symmetries.

1. $F = 0, A_1 = 0, A_2 = 0, B_1 = 0, C_1 = 0$. Canonical form of metric:

$$G = dx^3 \otimes dx^3 \otimes dx^3 + \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{S}(dx^2 \otimes dx^3 \otimes dx^3) + C_2 \hat{S}(dx^1 \otimes dx^1 \otimes dx^3) \quad (40)$$

$$+ C_3 \hat{S}(dx^2 \otimes dx^2 \otimes dx^3).$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 + 2C_3x^2)\partial_1 - (x^3 + 2C_2x^1)\partial_2.$$

2. $F = 0, A_1 = 0, B_1 = 0, C_1 = 0, C_2 = 0$. Canonical form of metric:

$$G = dx^2 \otimes dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \otimes dx^3 + \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) + B_3 \hat{S}(dx^2 \otimes dx^3 \otimes dx^3) \quad (41)$$

$$+ C_3 \hat{S}(dx^2 \otimes dx^2 \otimes dx^3).$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = ((B_3 - 1)x^3 + 2(C_3^2 - B_3)x^2 - 2x^1)\partial_1 - C_3x^3\partial_2 + x^3\partial_3.$$

3. $A_1 = 0, A_2 = 0, A_3 = 0, B_1 = 0, C_1 = 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^3) + C_2 \hat{S}(dx^1 \otimes dx^1 \otimes dx^3) + C_3 \hat{S}(dx^2 \otimes dx^2 \otimes dx^3) + \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) \quad (42)$$

$$+ \hat{S}(dx^2 \otimes dx^3 \otimes dx^3).$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 + 2x^1 + 2C_3x^2)\partial_1 - (2x^2 + 2C_2x^1 + x^3)\partial_2.$$

4. $A_1 = 0, A_2 = 0, B_1 = 0, B_2 = 0, C_1 = 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^3) + C_2 \hat{S}(dx^1 \otimes dx^1 \otimes dx^3) + C_3 \hat{S}(dx^2 \otimes dx^2 \otimes dx^3) + \hat{S}(dx^2 \otimes dx^3 \otimes dx^3) + \quad (43)$$

$$+ dx^3 \otimes dx^3 \otimes dx^3.$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^1 + x^2/C_2)\partial_2 - \frac{2x^1 + 2C_3x^2 + x^3}{2C_2}\partial_1.$$

5. $A_1 = 0, A_2 = 0, B_1 = 0, C_1 = 0, C_2 = 0$. Canonical form of metric:

$$G = \hat{S}(dx^1 \otimes dx^2 \otimes dx^3) + dx^3 \otimes dx^3 \otimes dx^3 + \hat{S}(dx^1 \otimes dx^3 \otimes dx^3) + C_3 \hat{S}(dx^2 \otimes dx^2 \otimes dx^3) \quad (44)$$

$$+ B_3 \hat{S}(dx^2 \otimes dx^3 \otimes dx^3).$$

Algebra of nontrivial symmetries is 1-dimensional:

$$X = (x^3 + 2x^2)\partial_2 - (2x^1 + 2C_3x^2 + B_3x^3)\partial_1.$$

3.6 Metrics with $\tau(G) = 6$ (1 type)

There exists the only metric of general kind: $A_1 = 0, A_2 = 0, B_1 = 0, C_1 = 0$:

$$G = F\hat{\mathcal{S}}(dx^1 \otimes dx^2 \otimes dx^3) + \hat{\mathcal{S}}(dx^1 \otimes dx^3 \otimes dx^3) + \hat{\mathcal{S}}(dx^2 \otimes dx^3 \otimes dx^3) + dx^3 \otimes dx^3 \otimes dx^3 \quad (45)$$

$$C_2\hat{\mathcal{S}}(dx^1 \otimes dx^1 \otimes dx^3) + C_3\hat{\mathcal{S}}(dx^2 \otimes dx^2 \otimes dx^3).$$

with 1-dimensional algebra of nontrivial symmetries:

$$X = (x^3 + 2Fx^1 + 2C_3x^2)\partial_1 - (2Fx^2 + 2C_2x^1 + x^3)\partial_2.$$

3.7 Metrics with $\tau(G) = 7, 8, 9, 10$.

Among the metrics of these affine types there is no metrics with nontrivial symmetries.

So there are 41 general cubic homogeneous metrics of different affine types, possessing nontrivial isometries. Note, that our analysis deals only with general affine types. Some general affine types with trivial isometries may contain special metrics with some relations on its components, possessing nontrivial isometries. In majority cases such isometries will be equivalent to one of the isometries, associated with considered metrics with $\tau(G) \leq 6$. In these cases we have equivalent metrics. However, it is possible the situation when under some particular values of metric components the metric will not be equivalent to any of above considered ones. Such "very special" metrics come out from the scope of our investigation (see, however, the table at the end of the section 5).

4 Affine-invariant classification

Some of the affine types with nontrivial symmetries are, in fact, affine-equivalent. In order to clear the question on equivalence of the above listed 41 classes, let us turn to (affine-)invariant properties of their symmetries fields. Preliminary classifying can be carried out by dimension of symmetries algebra. Combining different affine types possessing equal dimensions of symmetry algebra, we go to the following non-equivalent classes:

1. class of affine types with 2-dimensional algebra of symmetry, including the cases (first number is affine type, second number is order number in correspondent section): 1.1, 2.2, 2.5, 2.6, 3.3, 3.5, 3.9, 3.13, 4.9;
2. class of affine types with 1-dimensional algebra of symmetry, including the cases: 2.1, 2.3, 2.4, 3.1, 3.2, 3.4, 3.6, 3.7, 3.8, 3.10, 3.11, 3.12, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.10, 5.1, 5.2, 5.3, 5.4, 5.5, 6.1.
3. types 1.2, 1.3, 2.7, 2.8, 2.9 with infinitely-dimensional algebra of symmetries;
4. all types, without nontrivial symmetries;
5. "very special metrics", which have not been included in previous items.

The two last classes come out of the scope of our investigation. The first two classes admit further more detailed classifying. Direct calculation shows, that commutators of the pair of symmetry fields for metrics of the first class are:

1. 0, for the cases 1.1, 2.2, 2.5, 3.3, 3.5;
2. $(3/2)X_2$ for the cases 2.6, 3.9, 3.13, 4.9.

So, we conclude, that *groups of metrics* $\{1.1, 2.2, 2.5, 3.3, 3.5\}$, and $\{2.6, 3.9, 3.13, 4.9\}$ are *affine-nonequivalent*. The question on affine equivalency of metrics inside these groups remains opened. We come back to this question in next section.

Let us go to the affine types with 1-dimensional algebras of symmetry. Rough classifying of these types may be carried out by comparing of the simplest affine invariant of their algebras — divergencies of corresponding vector fields: $\text{div } X \equiv \partial_i X^i$. Elementary calculations show, that $\text{div } X = 0$ for the following cases: 2.1, 2.5, 3.1, 3.2, 3.4, 3.6, 3.8, 3.11, 3.12, 4.1, 4.2, 4.3, 4.4, 4.7, 4.10, 5.1, 5.3, 5.4, 5.5, 6.1 and $\text{div } X = \text{const} \neq 0$ for the cases 2.3, 2.4, 3.7, 3.10, 4.5, 4.6, 4.8, 5.2. So, *affine types, lying in these different groups, are affine-nonequivalent*.

Further more detailed classifying of the metrics inside these groups implies comparing of other affine invariants. Since all considered symmetries are described by linear vector fields, let us consider the following *matrix of vector field* A , defined by relation:

$$X^\alpha = A_\beta^\alpha x^\beta,$$

where A_β^α — components of A . This definition means, that A is affine tensor of valency $(1, 1)$. Its affine invariants are the following quantities:

$$I_1 \equiv \text{Tr}(A), \quad \dots, I_n \equiv \text{Tr}(A^n); \quad \Delta \equiv \det(A).$$

Note, that $\text{div } X = I_1$. Equivalent metrics must satisfy *colinearity conditions*:

$$\sqrt[n]{\frac{I_n}{I'_n}} = \sqrt[3]{\frac{\Delta}{\Delta'}} = C = \text{const} \quad (46)$$

for all $n = 1, \dots$, where $\{I_n, \Delta\}$ is the system of invariants of one metric, $\{I'_n, \Delta'\}$ is the system of invariants for another one. It is possible to construct other invariants, but the set $\{I_n\}$ is sufficient for our purposes.

For the metrics with $\text{div } X \neq 0$ matrices of their vector fields and invariants have the following kind:

$$\begin{aligned} 1.2 : \quad & \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_n = 1 + (-2)^n; \quad 2.3 : \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, I_n = \frac{1 + (-2)^n}{(-2)^n}; \\ 2.4 : \quad & \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_n = 1 + (-2)^n; \quad 3.7 : \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{pmatrix}, I_n = 1 + (-2)^n; \\ 3.10 : \quad & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -2\epsilon & 0 & -2 \end{pmatrix}, I_n = 1 + (-2)^n; \quad 4.5 : \quad \begin{pmatrix} 1 & 0 & 0 \\ -B & 0 & 0 \\ 0 & 2B^2 & -2 \end{pmatrix}, I_n = 1 + (-2)^n; \\ 4.6 : \quad & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2B & 1 & -2 \end{pmatrix}, I_n = 1 + (-2)^n; \quad 2.3 : \quad \begin{pmatrix} 0 & 0 & -C \\ 2(C^2 \mp 1) & -2 & \pm C \\ 0 & 0 & 1 \end{pmatrix}, I_n = 1 + (-2)^n; \\ 5.2 : \quad & \begin{pmatrix} -2 & 2(C_3^2 - B_3) & B_3 C_3 - 1 \\ 0 & 0 & -C_3 \\ 0 & 0 & 1 \end{pmatrix}, I_n = 1 + (-2)^n. \end{aligned}$$

Obviously, that conditions (46) are satisfied for all metrics from the group with $\text{div } X = I_1 \neq 0$.

For the group with $\text{div } X = I_1 = 0$ matrices of their vector fields and invariants have the following kind:

$$2.1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, I_n = 1 + (-1)^n; \quad 3.1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, I_n = 1 + (-1)^n;$$

$$3.2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & -1 \end{pmatrix}, I_n = 1 + (-1)^n; \quad 3.4 : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1/2 & -1 \end{pmatrix}, I_n = 1 + (-1)^n;$$

$$3.6 : \begin{pmatrix} -1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, I_n = 1 + (-1)^n;$$

$$3.8 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\epsilon_1 \epsilon_2 & 0 \end{pmatrix}, I_n = (-\epsilon_1 \epsilon_2)^{n/2} (1 + (-1)^n);$$

$$3.11 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \mp 1/2 & \mp 1 & 0 \end{pmatrix}, I_n = (\mp 1)^{n/2} (1 + (-1)^n);$$

$$3.12 : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & -1/2 & 0 \end{pmatrix}, I_n = (-1)^{n/2} (1 + (-1)^n);$$

$$4.1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon_2 F & 1 \\ 0 & -\epsilon_1 \epsilon_2 & -\epsilon_2 F \end{pmatrix}, I_n = (F^2 - \epsilon_1 \epsilon_2)^{n/2} (1 + (-1)^n);$$

$$4.2 : \begin{pmatrix} 0 & 0 & 0 \\ -1/2F & -1 & 0 \\ 0 & \pm 1/F & 1 \end{pmatrix}, I_n = (1 + (-1)^n);$$

$$4.3 : \begin{pmatrix} 0 & 0 & 0 \\ \mp 1/2 & \mp F & \mp 1 \\ 0 & 1 & \pm F \end{pmatrix}, I_n = (F^2 \mp 1)^{n/2} (1 + (-1)^n);$$

$$4.4 : \begin{pmatrix} 1 & 0 & 1/2F \\ -1/F & -1 & -1/2F \\ 0 & 0 & 0 \end{pmatrix}, I_n = 1 + (-1)^n;$$

$$4.7 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\epsilon_2 C/2 & -\epsilon_1 \epsilon_2 & 0 \end{pmatrix}, I_n = (-\epsilon_1 \epsilon_2)^{n/2} (1 + (-1)^n);$$

$$4.10 : \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1 \\ -1/2 & -B & 0 \end{pmatrix}, I_n = (-B)^{n/2} (1 + (-1)^n);$$

$$5.1 : \begin{pmatrix} 0 & 2C_3 & 1 \\ -2C_2 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, I_n = (-4C_2 C_3)^{n/2} (1 + (-1)^n);$$

$$5.3 : \begin{pmatrix} 2 & 2C_3 & 1 \\ -2C_2 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}, I_n = (4(1 - C_2 C_3))^{n/2} ((-1)^n + 1);$$

$$\begin{aligned}
5.4 \quad & \begin{pmatrix} -1/C_2 & -C_3/C_2 & -1/2C_2 \\ 1 & 1/C_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_n = (1 - C_2C_3)^{n/2}(1 + (-1)^n); \\
5.5 : \quad & \begin{pmatrix} -2 & -2C_3 & -B_3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_n = 4^{n/2}(1 + (-1)^n); \\
6.1 \quad & \begin{pmatrix} 2F & 2C_3 & -B_3 \\ -2C_2 & -2F & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_n = 2^n(F^2 - C_2C_3)^{n/2}(1 + (-1)^n)
\end{aligned}$$

Comparison of the series of invariants leads to the following potential classes of affine equivalent metrics:

1. {2.1, 3.1, 3.2, 3.4, 3.6, 3.8 ($\epsilon_1\epsilon_2 < 0$), 3.11 (− in the metric), 4.1 ($F^2 > \epsilon_1\epsilon_2$), 4.2, 4.3 ($F^2 > \pm 1$), 4.4, 4.7 ($\epsilon_1\epsilon_2 < 0$), 4.10 ($B < 0$), 5.1 ($C_2C_3 < 0$), 5.3 $C_2C_3 < 1$, 5.4 ($C_2C_3 < 1$), 5.5, 6.1 ($F^2 > C_2C_3$)};
2. {3.8 ($\epsilon_1\epsilon_2 > 0$), 3.11 (+ in the metric), 3.12, 4.1 ($F^2 < \epsilon_1\epsilon_2$), 4.3 ($F^2 < \pm 1$), 4.7 ($\epsilon_1\epsilon_2 > 0$), 4.10 ($B < 0$), 5.1 ($C_2C_3 > 0$), 5.3 $C_2C_3 > 1$, 5.4 ($C_2C_3 < 1$), 6.1 ($F^2 < C_2C_3$)};
3. {4.1 ($F = \epsilon_1\epsilon_2$), 4.3 ($F^2 = +1$) ("+" in the metric), 4.10 $B = 0$, 5.3 $C_2C_3 = 1$, 5.4 ($C_2C_3 = 1$), 6.1 ($F^2 = C_2C_3$)};

The more detailed additional investigation of the cases 3, when all invariants formally vanish, gives the following corrections:

1. The metric 4.1 under $F^2 = \epsilon_1\epsilon_2$ admits symmetry vector field with one arbitrary function of all coordinates, i.e. admits infinitely-dimensional group of symmetry.
2. The metrics 4.3 under $F^2 = 1$, 4.10 under $B = 0$, 5.3, 5.4 under $C_2C_3 = 1$ and 6.1 under $F^2 = C_2C_3$ admit 2-dimensional nonabelian group of symmetries with commutator of the kind: $[X_1, X_2] = (3/2)X_2$.

Resuming our investigation, we conclude, that *all homogeneous cubic metrics of general affine types are divided on 8 affine-nonequivalent classes*:

1. class {1.1, 2.2, 2.5, 3.3, 3.5} (2-dimensional abelian algebra of nontrivial symmetries);
2. class {2.6, 3.9, 3.13, 4.9, 4.3 ($F^2 = 1$), 4.10 ($B = 0$), 5.3, 5.4 ($C_2C_3 = 1$), 6.1 ($F^2 = C_2C_3$)} (2-dimensional nonabelian algebra of nontrivial symmetries);
3. class {1.2, 1.3, 2.7, 2.8, 2.9, 4.1 ($F^2 = \epsilon_1\epsilon_2$)} (infinitely-dimensional algebra of isometries); one can subdivide this class on the following subclasses: (1): ∞^2 -dimensional group (1.3), (2): ∞ -dimensional group (4.1, 2.8, 2.9) and (3): $\infty + 1$ -dimensional group (1.2, 2.7);
4. class {2.3, 2.4, 3.7, 3.10, 4.5, 4.6, 4.8, 5.2} (1-dimensional algebra of nontrivial symmetries with $I_1 \neq 0$);
5. class {2.1, 3.1, 3.2, 3.4, 3.6, 3.8 ($\epsilon_1\epsilon_2 < 0$), 3.11 (− in metric), 4.1 ($F^2 > \epsilon_1\epsilon_2$), 4.2, 4.3 ($F^2 > \pm 1$), 4.4, 4.7 ($\epsilon_1\epsilon_2 < 0$), 4.10 ($B < 0$), 5.1 ($C_2C_3 < 0$), 5.4 ($C_2C_3 < 1$), 5.5, 6.1 ($F^2 > C_2C_3$)} (1-dimensional algebra of nontrivial symmetries, $I_n = C^{n/2}(1 + (-1)^n)$, $C = \text{const} > 0$);

6. class {3.8 ($\epsilon_1\epsilon_2 > 0$), 3.11 (+ in metric), 3.12, 4.1 ($F^2 < \epsilon_1\epsilon_2$), 4.3 ($F^2 < \pm 1$), 4.7 ($\epsilon_1\epsilon_2 > 0$), 4.10 ($B < 0$), 5.1 ($C_2C_3 > 0$), 5.4 ($C_2C_3 < 1$), 6.1 ($F^2 < C_2C_3$)}; (1-dimensional algebra of nontrivial symmetries, $I_n = (-C)^{n/2}(1 + (-1)^n)$, $C = \text{const} < 0$);
7. class of metrics with nontrivial symmetries, which are absent in previous list;
8. class of metrics without symmetries.

The question on affine equivalence of the metrics inside these classes is opened. Next section we'll prove that the answer is, generally speaking, negative.

5 Connection with projective classification

Lets clear connection of obtained results with well known projective classification of cubic 3-dimensional forms [10]. Combination of methods of projective geometry and cubic matrix algebra leads to the following classifying theorem.

THEOREM (on classification of real cubic forms) Any cubic form over field of real numbers belong to one of the classes of real affine-equivalency (only nonzero components of canonical kind of cubic metric are presented):

1. general class $A_1 = A_2 = A_3 = 1$, with 10 nonequivalent subclasses: $F < -(\sqrt{3} + 1)/2$, $F = -(\sqrt{3} + 1)/2$, $-(\sqrt{3} + 1)/2 < F < -1/2$, $-1/2 < F < 0$, $F = 0$, $0 < F < (\sqrt{3} - 1)/2$, $F = (\sqrt{3} - 1)/2$, $(\sqrt{3} - 1)/2 < F < 1$, $F = 1$, $F > 1$.
2. Degenerated class I: $A_1 = A_2 = F = 1$;
3. Degenerated class II: $A_1 = F = 1$;
4. Degenerated class III: $F = 1$;
5. Degenerated class IV: $A_1 = C_3 = 1$;
6. Degenerated class V: $C_1 = C_3 = 1$;
7. Degenerated class VI: $A_1 = A_2 = 1$;
8. Degenerated class VII: $C_1 = 1$;
9. Degenerated class VIII: $A_1 = 1$;
10. Degenerated class IX: $A_3 = C_1 = B_3 = 1$;
11. Degenerated class X: $-A_2 = C_1 = B_3 = 1$;
12. Degenerated class XI: $A_2 = C_1 = B_3 = 1$;
13. Degenerated class XII: $C_1 = B_3 = 1$;
14. Degenerated class XI: $-A_2 = C_1 = 1$.

Comparison of these canonical types with classes of isometries leads to the following conclusions:

1. General class under $F \neq -1/2$ has no nontrivial symmetries and so it belongs to symmetry class 8. In case $F = -1/2$ generic metric acquires 2-dimensional abelian group of symmetries and can be related to the symmetry class 1;

2. Degenerated class I has no nontrivial symmetries and so it belongs to symmetry class 8;
3. Degenerated class II has 1-dimensional group with $I_1 = 0$ and is related to symmetry class 5;
4. Degenerated class III has 2-dimensional abelian group and is related to symmetry class 1;
5. Degenerated class IV has 1-dimensional group with $I_1 \neq 0$ and is related to symmetry class 4;
6. Degenerated class V has 2-dimensional nonabelian group with is related to symmetry class 2;
7. Degenerated class VI has ∞ -dimensional group and is related to symmetry class 3(2);
8. Degenerated class VII has $\infty + 1$ -dimensional group and is related to symmetry class 3(3)
9. Degenerated class VIII has ∞ -dimensional group and is related to symmetry class 3(1);
10. Degenerated class IX has no nontrivial symmetries and is related to symmetry class 9;
11. Degenerated class X has 1-dimensional group with $I_1 = 0$ and is related to symmetry class 5;
12. Degenerated class XI has 1-dimensional group with $I_1 = 0$ and is related to symmetry class 5;
13. Degenerated class XII has 2-dimensional abelian group and is related to symmetry class 1;
14. Degenerated class XIII has ∞ -dimensional group and is related to symmetry class 3(2).

Interrelations between symmetry and projective classifications are resumed in the following table.

| Symmetries classes | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|--------------------|---------|---|-----------------------------------|----|---------|---|---|-----------|
| Projective classes | III,XII | V | (1): VIII, (2): VI,XIII, (3): VII | IV | II,X,XI | ? | — | Gen, I,IX |

Analysis of the table leads to the following important conclusions:

1. Symmetries classification is more rough, then projective, since some classes of symmetries contain several non-equivalent projective classes.
2. Emptiness of the column with number 7 means, that we have studied in fact all non-equivalent classes of cubic metrics.
3. Emptiness of the column with number 6 means that 5-th and 6-th symmetries classes are identical. Common constant in righthand side of the colinearity condition (46) between these classes will be imaginary. This corresponds to the statement, that isometries fields form not R -module, as we have assumed, but C -module.

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